J. Math. Anal. Appl. 356 (2009) 201–207

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

An algebroid function and its derivative ☆

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ARTICLE INFO

Article history:
Received 28 October 2008
Available online 9 March 2009
Submitted by A.V. Isaev

Keywords:
Algebroid function
Derivative
Characteristic function

ABSTRACT

In this paper, we investigate the value distribution of an algebroid function and its derivative, and obtain two inequations between Nevanlinna characteristic function of an algebroid function and that of its derivative. We extend Chuang Chitai's theorem of meromorphic functions to algebroid functions.

1. Introduction

Although the value distribution theory of meromorphic functions due to R. Nevanlinna (see [1,2] for standard references) was extended to the corresponding theory of algebroid functions by H. Selberg [3,4], E. Ullrich [5] and G. Valiron [6] around 1930, up to now, lots of important theorems on meromorphic functions have not been extended to algebroid functions for their multivaluedness and the complexity of their branch points. In the present paper, we study the relation between the growth of algebroid function and that of its derivative which extends Chuang Chitai's theorem of meromorphic functions to algebroid functions. Finally, we testify that the order and the lower order of an algebroid function are equal to that of its derivative respectively.

Suppose $A_k(z), \dots, A_0(z)$ are analytic functions with no common zeros on the complex plane $\mathbb{C}$, then bivariate complex function

$$\Phi(z,W) = A_v(z)W^v + A_{v-1}(z)W^{v-1} + \cdots + A_1(z)W + A_0(z) = 0 \tag{1.1}$$

defines a v-valued algebroid function $W(z)$ on the complex plane [7,8]. If $A_v(z) = 1$, then $W(z)$ is called v-valued entire algebroid function. If $A_v(z), \dots, A_0(z)$ are all polynomial functions, then $W(z)$ is called v-valued algebraic function. In addition, if some of $A_v(z), A_{v-1}(z), \dots, A_0 z$ are transcendental entire functions, then $W(z)$ is said to be a transcendental algebroid function. The bivariate complex function (1.1) can also be written as

$$\Phi(z,W) = A_v(z)\big(W - w_1(z)\big)\big(W - w_2(z)\big)\cdots\big(W - w_v(z)\big) = 0,$$

where $w_i(z)$ $(i = 1, 2, \dots, v)$ are the branches of $W(z)$ and they are all meromorphic functions.

In this paper, unless otherwise stated, algebroid functions are all transcendental on the complex plane. We use the standard notation of Nevanlinna's value distribution theory and assume that the reader is familiar with the basic results of value distribution theory of algebroid functions (see e.g. [7]).

☆ Supported by the National Natural Science Foundation of China (Grant No. 10271011) and the National Key Basic Research Project of China (2005CB321902).

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.03.014

2. Some lemmas

Definition 2.1. Suppose that $W(z)$ is a v-valued algebroid function defined by (1.1), the modulus of $W(z)$ is defined by

$$\left|W(z)\right| = \max_{1\leqslant j\leqslant v}\left\{\left|w_j(z)\right|\right\}$$

when z is not a pole of $W(z)$; $|W(z)| = \infty$ when z is a pole of $W(z)$.

Definition 2.2. Suppose that algebroid function $W(z)$ has no poles in $|z| < r'$, then for $r < r'$, we define the modulus function of $W(z)$ in $|z| < r$ as

$$M(r, W) = \max_{1\leqslant j\leqslant v}\sup\left\{\left|w_j(z)\right|;\ |z| \leqslant r\right\}.$$

Lemma 2.1. *(See [9].) Let z_j $(j = 1, 2, \ldots, n)$ be n different complex numbers, none of which is zero, then there exists a real number θ_0 such that*

$$\left|\prod_{j=1}^{n}\left(te^{i\theta_0} - z_j\right)\right| > \frac{1}{2^n}\prod_{j=1}^{n}|z_j|$$

holds for any real number t.

Lemma 2.2. *Suppose that $W(z)$ is a v-valued algebroid function defined by* (1.1)*, satisfying that $\infty \notin \{W(0)\}$, and R, R' are two positive numbers, $R < R'$. Then there exists a real number θ_0 such that*

$$\log^+\left|W\left(te^{i\theta_0}\right)\right| \leqslant n(R', W)\log 4 + vN(R', W) + v\cdot\frac{R'+R}{R'-R}m(R', W)$$

holds for $0 \leqslant t \leqslant R$.

Proof. Let z_j $(j = 1, 2, \ldots, n)$ be the poles of $W(z)$ in the disk $|z| \leqslant R'$, with due count of multiplicity, and let

$$p(z) = \frac{1}{(2R')^n}\prod_{j=1}^{n}(z - z_j), \qquad G(z) = W(z)p(z).$$

Thus $G(z)$ has no poles in $|z| < R'$. Especially, if $A_v(z)$ is a polynomial, then $G(z)$ is a v-valued entire algebroid function, because the poles of $W(z)$ are the zeros of $A_v(z)$ with the same multiple numbers. So

$$\begin{aligned}\Psi(z, G) &= B_v(z)G^v + B_{v-1}(z)G^{v-1} + \cdots + B_1(z)G + B_0(z)\\ &= B_v(z)\left(G - w_1(z)p(z)\right)\left(G - w_2(z)p(z)\right)\cdots\left(G - w_v(z)p(z)\right) = 0,\end{aligned}$$

where $B_v(z)$ has no zeros in $|z| \leqslant R'$. Especially, $B_v(z) \equiv 1$ when $A_v(z)$ is a polynomial. Then $G(z)$ is finite and satisfies $|G(z)| \leqslant |W(z)|$ in $|z| \leqslant R'$.

By Lemma 2.1, there exists a real number θ_0 such that

$$\left|\prod_{j=1}^{n}\left(te^{i\theta_0} - z_j\right)\right| > \frac{1}{2^n}\prod_{j=1}^{n}|z_j|$$

holds for any real number t. Therefore, if $0 \leqslant t \leqslant R$, we have

$$\left|G\left(te^{i\theta_0}\right)\right| \geqslant \left|W\left(te^{i\theta_0}\right)\right|\frac{1}{(2R')^n}\cdot\frac{1}{2^n}\prod_{j=1}^{n}|z_j|.$$

So

$$\left|W\left(te^{i\theta_0}\right)\right| \leqslant \frac{4^n R'^n}{\prod_{j=1}^{n}|z_j|}\left|G\left(te^{i\theta_0}\right)\right|,$$

and then

$$\log^+\left|W\left(te^{i\theta_0}\right)\right| \leqslant n(R', W)\log 4 + vN(R', W) + \log^+ M(R, G).$$

Since

$$\log^+ M(R,G) \leqslant v\frac{R'+R}{R'-R}m(R',G) \leqslant v\cdot\frac{R'+R}{R'-R}m(R',W),$$

thus

$$\log^+ \left|W\left(te^{i\theta_0}\right)\right| \leqslant n(R',W)\log 4 + v\cdot N(R',W) + v\cdot\frac{R'+R}{R'-R}m(R',W),$$

which completes the proof. □

Lemma 2.3. *(See [9].) Suppose z_j $(j=1,2,\ldots,n)$ are n different complex numbers and R, R' are two positive numbers, satisfying $R<R'$, then there exists a positive number ρ $(R\leqslant\rho\leqslant R')$ such that*

$$\prod_{j=1}^{n}(z-z_j) > \frac{(R'-R)^n}{(2e)^n}$$

for $|z|=\rho$.

Lemma 2.4. *Suppose $W(z)$ is a v-valued algebroid function algebroid function defined by (1.1) and R, R', R'' are three positive numbers, satisfying $R<R'<R''$, then there exists a positive number ρ $(R\leqslant\rho\leqslant R')$ such that*

$$\log^+\left|W(z)\right| \leqslant n(R'',W)\log\frac{4eR''}{R'-R} + v\cdot\frac{R''+R'}{R''-R'}m(R'',W)$$

for $|z|=\rho$.

Proof. Let z_j $(j=1,2,\ldots,n)$ be the poles of $W(z)$ in $|z|\leqslant R''$, with due count of multiplicity, and let

$$p(z)=\frac{1}{(2R'')^n}\prod_{j=1}^{n}(z-z_j),\qquad G(z)=W(z)p(z).$$

It is same as the proof of Lemma 2.2, we can get $G(z)$ is finite and satisfies $|G(z)|\leqslant|W(z)|$ in the circle $|z|=\rho$.

By Lemma 2.3, it follows that there exists a positive number ρ $(R\leqslant\rho\leqslant R')$ such that

$$\left|\prod_{j=1}^{n}(z-z_j)\right| > \frac{(R'-R)^n}{(2e)^n}$$

holds for $|z|=\rho$.

Thus we have

$$\left|G(z)\right| \geqslant \left|W(z)\right|\cdot\frac{1}{(2R'')^n}\cdot\frac{(R'-R)^n}{(2e)^n},$$

$$\left|W(z)\right| \leqslant \frac{(4eR'')^n}{(R'-R)^n}\left|G(z)\right|,$$

$$\log^+\left|W(z)\right| \leqslant n(R'',W)\log\frac{4eR''}{R'-R} + \log^+ M(R',G).$$

Since

$$\log^+ M(R',G) \leqslant v\cdot\frac{R''+R'}{R''-R'}m(R'',G) \leqslant v\cdot\frac{R''+R'}{R''-R'}m(R'',W),$$

we have

$$\log^+\left|W(z)\right| \leqslant n(R'',W)\log\frac{4eR''}{R''-R} + v\cdot\frac{R''+R'}{R''-R'}m(R'',W). \qquad \square$$

Lemma 2.5. *(See [7].) Suppose $W(z)$ is an algebroid function defined by (1.1), then*

$$N(r,W)+k\overline{N}(r,W) \leqslant N\left(r,W^{(k)}\right) \leqslant N(r,W)+k\overline{N}(r,W)+(2k-1)N_x(r,W)+O(1),$$

where $N_x(r,W)$ is counting function of branch points of $W(z)$.

3. Theorems and proofs

Theorem 3.1. *Suppose $W(z)$ is a v-valued algebroid function defined by* (1.1)*, satisfying $\infty \notin \{W(0)\}$, then*

$$T(r, W') < 12 \cdot \frac{\lambda}{\lambda - 1} \cdot vT(\lambda r, W) + 2\log^+ \frac{1}{r} + 2\log^+ \frac{\lambda}{\lambda - 1} + 5 \tag{3.2}$$

and

$$T(r, W) < 40v \frac{\lambda}{\lambda - 1} \log \frac{e\lambda}{\lambda - 1} T(\lambda r, W') + \log^+(\lambda r) + 5 + \log^+ \left|W(0)\right| \tag{3.3}$$

hold for $\lambda > 1$ and $r > 0$.

Proof. Let $k = \lambda^{\frac{1}{3}}$, $r_1 = kr$, $r_2 = k^2 r$, $r_3 = k^3 r$. The poles of $W(z)$ in $|z| \leqslant r_2$ denoted by z_j $(j = 1, 2, \dots, n)$, with due count of multiplicity, and let

$$p(z) = \frac{1}{(2r_2)^n} \prod_{j=1}^{n} (z - z_j), \qquad G(z) = W(z)p(z).$$

Likewise, we know that $G(z)$ is finite and satisfies that $|G(z)| \leqslant |W(z)|$ in $|z| \leqslant r_2$.

According to the identical equation

$$W'(z) = \frac{1}{p(z)} p(z)W'(z), \qquad p(z)W'(z) = G'(z) - W(z)p'(z),$$

we have

$$\begin{aligned} m(r, W') &\leqslant m\left(r, \frac{1}{p}\right) + m(r, pW') \\ &\leqslant m\left(r, \frac{1}{p}\right) + m(r, G') + m(r, W) + m(r, p') + \log 2. \end{aligned} \tag{3.4}$$

Since

$$m(r, p) + N(r, p) = m\left(r, \frac{1}{p}\right) + N\left(r, \frac{1}{p}\right) + \log\left|p(0)\right|,$$

we have

$$\begin{aligned} m\left(r, \frac{1}{p}\right) &\leqslant \log \frac{1}{|p(0)|} = \log \frac{(2r_2)^n}{|z_1 z_2 \cdots z_n|} \\ &= n\log(2r_2) + \log \frac{1}{|z_1 z_2 \cdots z_n|} \\ &= n\log 2 + n\log r_2 + \log \frac{1}{|z_1 z_2 \cdots z_n|} \\ &= n(r_2, W)\log 2 + vN(r_2, W). \end{aligned} \tag{3.5}$$

From

$$M(r, p') \leqslant \frac{1}{r_1 - r} M(r_1, p) \leqslant \frac{1}{r_1 - r}, \tag{3.6}$$

we obtain

$$m(r, p') \leqslant \log^+ M(r, p') \leqslant \log^+ \frac{1}{r_1 - r}. \tag{3.7}$$

On the other hand,

$$M(r, G') \leqslant \frac{1}{r_1 - r} M(r, G)$$

and

$$\log^+ M(r_1, G) \leqslant v \cdot \frac{r_2 + r_1}{r_2 - r_1} m(r_2, G) \leqslant v \cdot \frac{r_2 + r_1}{r_2 - r_1} m(r_2, W)$$

imply that

$$\begin{aligned} m(r,G') &\leqslant \log^+ M(r,G') \\ &\leqslant \log^+ M(r_1,G) + \log^+ \frac{1}{r_1-r} \\ &\leqslant v\cdot\frac{r_2+r_1}{r_2-r_1} m(r_2,W) + \log^+ \frac{1}{r_1-r}. \end{aligned} \tag{3.8}$$

Hence by (3.4)–(3.8),

$$m(r,W') \leqslant n(r_2,W)\log 2 + vN(r_2,W) + v\cdot\frac{r_2+r_1}{r_2-r_1} m(r_2,W) + m(r,W) + 2\log^+ \frac{1}{r_1-r} + \log 2. \tag{3.9}$$

By Lemma 2.5, we have

$$\begin{aligned} N(r,W') &\leqslant N(r,W) + \overline{N}(r,W) + N_x(r,W) + O(1) \\ &\leqslant 2N(r,W) + N_x(r,W) + O(1). \end{aligned} \tag{3.10}$$

In addition, it is obvious that

$$n(r_2,W)\log\frac{r_3}{r_2} \leqslant v\cdot N(r_3,W) \leqslant v\cdot T(r_3,W), \tag{3.11}$$

$$\begin{aligned} v\cdot N(r_2,W) + v\cdot\frac{r_2+r_1}{r_2-r_1} m(r_2,W) &\leqslant v\cdot\frac{r_2+r_1}{r_2-r_1} T(r_2,W) \\ &\leqslant v\cdot\frac{r_2+r_1}{r_2-r_1} T(r_3,W), \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} m(r,W) + 2N(r,W) + N_x(r,W) + O(1) &\leqslant 2T(r,W) + 2(v-1)T(r,W) + O(1) \\ &= 2vT(r,W) + O(1). \end{aligned} \tag{3.13}$$

By (3.9)–(3.13), we obtain

$$T(r,W') \leqslant \left(\frac{\log 2}{\log\frac{r_3}{r_2}} + \frac{r_2+r_1}{r_2-r_1} + 2\right)\cdot v\cdot T(r_3,W) + 2\log^+\frac{1}{r_1-r} + \log 2. \tag{3.14}$$

Finally, by (3.14) and

$$\frac{1}{k-1} < \frac{1}{\log k} = \frac{3}{\log\lambda} < \frac{3\lambda}{\lambda-1},$$

we get

$$T(r,W') < 12\cdot\frac{\lambda}{\lambda-1}\cdot vT(\lambda r,W) + 2\log^+\frac{1}{r} + 2\log^+\frac{\lambda}{\lambda-1} + 5.$$

Thus, we obtain (3.2). Next, we will show that (3.3) is true.

Consider the v-valued algebroid function $W'(z)$ and positive numbers $k=\lambda^{\frac{1}{3}}$, $r_1=kr$, $r_2=k^2r$, $r_3=k^3r$.

According to Lemma 2.2, there exists a real number θ_0 such that

$$\log^+\left|W'\left(te^{i\theta_0}\right)\right| \leqslant n(r_2,W')\log 4 + vN(r_2,W') + v\frac{r_2+r_1}{r_2-r_1} m(r_2,W') \tag{3.15}$$

holds for $0\leqslant t\leqslant r_1$.

According to Lemma 2.4, there exists a positive number ρ $(r\leqslant\rho\leqslant r_1)$ such that

$$\log^+\left|W'(z)\right| \leqslant n(r_2,W')\log\frac{4er_2}{r_1-r} + \frac{r_2+r_1}{r_2-r_1}\cdot v\cdot m(r_2,W') \tag{3.16}$$

holds for $|z|=\rho$.

Suppose z is a point on $|z|=\rho$ and h is a real number satisfying $z=\rho e^{i(\theta_0+h)}$, $0\leqslant h<2\pi$. It can be seen that

$$w_i(z) - w_i(0) = \int_{L_1} w_i'(z)\,dz + \int_{L_2} w_i(z)\,dz$$

for every i, where L_1 is the line segment $z=te^{i\theta_0}$ $(0\leqslant t\leqslant\rho)$ and L_2 is the arc $z=\rho e^{i\theta}$ $(\theta_0\leqslant\theta\leqslant\theta_0+h)$.

Let

$$M_1 = \max_{1\leqslant i\leqslant v}\max_{0\leqslant t\leqslant r_1}\{|w_i'(te^{i\theta_0})|\},\qquad M_2 = \max_{1\leqslant i\leqslant v}\max_{|z|=\rho}\{|w_i'(z)|\}.$$

Then

$$|w_i(z)-w_i(0)| \leqslant M_1\rho + M_2 h\rho \leqslant r_1(M_1+2\pi M_2),$$
$$|w_i(z)| \leqslant r_1(M_1+2\pi M_2) + |w_i(0)| \leqslant r_1(M_1+2\pi M_2) + |W(0)|.$$

It follows that

$$|W(z)| \leqslant r_1(M_1+2\pi M_2) + |W(0)|,$$
$$\log^+|W(z)| \leqslant \log^+ r_1 + \log^+ M_1 + \log^+ M_2 + \log^+|W(0)| + \log(8\pi).$$

By (3.15) and (3.16), we have

$$\log^+|W(z)| \leqslant n(r_2,W')\log\frac{16er_2}{r_1-r} + vN(r_2,W') + 2v\frac{r_2+r_1}{r_2-r_1}m(r_2,W') + \log^+ r_1 + \log(8\pi) + \log^+|W(0)|.$$

Hence

$$\log^+ M(\rho,W) \leqslant n(r_2,W')\log\frac{16er_2}{r_1-r} + vN(r_2,W') + 2v\frac{r_2+r_1}{r_2-r_1}m(r_2,W') + \log^+ r_1 + \log(8\pi) + \log^+|W(0)|. \tag{3.17}$$

It is easily seen that

$$m(\rho,W) \leqslant \log^+ M(\rho,W), \tag{3.18}$$
$$N(\rho,W) \leqslant N(\rho,W') \leqslant N(r_1,W') \leqslant N(r_2,W'), \tag{3.19}$$
$$n(r_2,W')\log\frac{r_3}{r_2} \leqslant vN(r_3,W') \leqslant vT(r_3,W'), \tag{3.20}$$
$$\begin{aligned}2vN(r_2,W') + 2v\frac{r_2+r_1}{r_2-r_1}m(r_2,W') &\leqslant 2v\cdot\frac{r_2+r_1}{r_2-r_1}T(r_2,W')\\ &\leqslant 2v\cdot\frac{r_2+r_1}{r_2-r_1}T(r_3,W'),\end{aligned} \tag{3.21}$$
$$T(r,W) \leqslant T(\rho,W). \tag{3.22}$$

By (3.17)–(3.22), we have

$$T(r,W) \leqslant \left(\frac{1}{\log\frac{r_3}{r_2}}\log\frac{16er_2}{r_1-r} + 2\frac{r_2+r_1}{r_2-r_1}\right)\cdot v\cdot T(r_3,W') + \log^+ r_1 + \log(8\pi) + \log^+|W(0)|. \tag{3.23}$$

Let C denote the coefficient of $T(r_3,W')$, that is,

$$C = \frac{v}{\log k}\log\frac{16ek^2}{k-1} + 2v\frac{k+1}{k-1}.$$

Since

$$\begin{aligned}\frac{v}{\log k}\log\frac{16ek^2}{k-1} &= \frac{v}{\log k}\left(\log(16e) + \log\frac{1}{k-1}\right) + 2v\\ &< \frac{v}{\log k}\left(\log(16e) + \log\frac{1}{\log k}\right) + 2v\\ &< \frac{3\lambda v}{\lambda-1}\left(\log(16e) + \log\frac{3\lambda}{\lambda-1}\right) + 2v,\end{aligned}$$

and

$$\frac{2v(k+1)}{k-1} = 2v\left(1+\frac{2}{k-1}\right) < 2v\left(1+\frac{6\lambda}{\lambda-1}\right),$$

we have

$$\begin{aligned}
C &< \frac{3\lambda v}{\lambda-1}\left(\log\frac{\lambda}{\lambda-1}+\log(16e)+\log 3+4\right)+4v\\
&< \frac{\lambda v}{\lambda-1}\log\frac{e\lambda}{\lambda-1}\left\{3\left[\log(16e)+\log 3+5\right]+4\right\}\\
&< 40v\frac{\lambda}{\lambda-1}\log\frac{e\lambda}{\lambda-1}.
\end{aligned}\tag{3.24}$$

From (3.23) and (3.24), we conclude that

$$T(r,W) < 40v\frac{\lambda}{\lambda-1}\log\frac{e\lambda}{\lambda-1}T(\lambda r,W') + \log^+(\lambda r) + 5 + \log^+\left|W(0)\right|,$$

which proves (3.3) is right. □

In Theorem 3.1, if we take $r_0 \geqslant 1$ such that

$$T(r,W) > 1, \qquad T(r,W') > \log r, \qquad T(r,W') > 1, \qquad T(r,W') > \log^+\left|W(0)\right|$$

hold for $r > r_0$, then we easily obtain the following:

Theorem 3.2. *Suppose $W(z)$ is a v-valued algebroid function defined by* (1)*, satisfying $\infty \notin \{W(0)\}$, then there exists a positive number r_0 such that*

$$T(r,W') < A\frac{\lambda v}{\lambda-1}T(\lambda r,W)$$

and

$$T(r,W) < B\frac{\lambda v}{\lambda-1}\log\frac{e\lambda}{\lambda-1}T(\lambda r,W')$$

hold for $\lambda > 1$ and $r > r_0$, where A, B are two positive absolute constants.

By the definition of order and lower order of algebroid functions, we directly obtain the following result.

Corollary 3.1. *Suppose $W(z)$ is a v-valued algebroid function defined by* (1)*, and $\infty \notin \{W(0)\}$, then $W(z)$ and $W'(z)$ have the same order and the same lower order.*

Acknowledgments

The author would like to thank Professor Zongsheng Gao, Professor Daochun Sun and the referees sincerely for their helpful suggestions and comments.

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